Sept 7, 2022

Week 1

1-dimensional Integration theory.

Let $[a, b]$ be the interval

$$\{x : a \leq x \leq b\}$$

and f be a function defined on $[a, b]$. Want to define

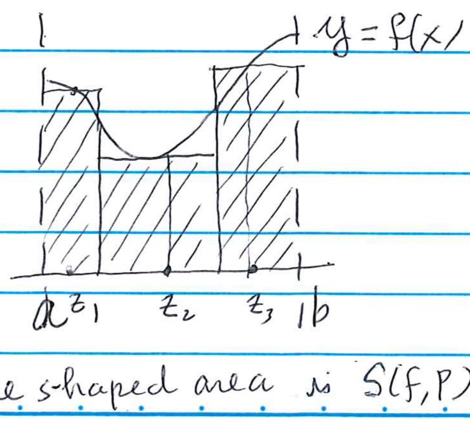
$$\int_a^b f(x)\, dx.$$

- A partition P is a collection of points
$$\{x_0, x_1, \cdots, x_n : x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

- A tag is $\{z_1, \cdots, z_n\}$ on P : $z_j \in [x_{j-1}, x_j]$, $j = 1, \cdots, n$.

- The Riemann sum of f with respect to the partition with tags is
$$S(f, P) = \sum_{j=1}^{n} f(z_j)\, \Delta x_j, \quad \Delta x_j = x_j - x_{j-1}.$$

When $f \geq 0$, $S(f, P)$ stands for an approximate area



the shaped area is $S(f, P)$

A function f is <u>integrable</u> if there is a number I such that as $\|P\|$ becomes small, the Riemann sum $S(f,P)$ comes close to I.

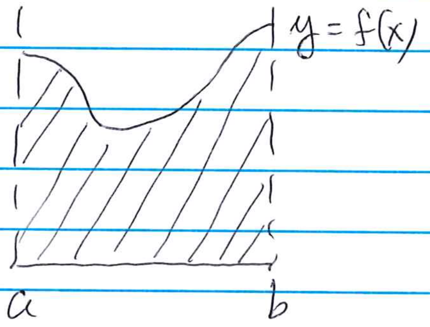
Here $\|P\| = \max\{\Delta x_1, \dots \Delta x_n\}$ is the <u>norm</u> of P.

The number I is called the <u>integral</u> of f over $[a,b]$.

When $f \geq 0$, it is understood as the area of the region bounded between the graph of f, the x-axis, and the vertical lines $x=a$, $x=b$.

Notation for I:

$$\int_a^b f, \quad \int_a^b f(x)\,dx, \quad \int_{[a,b]} f, \quad \text{etc}$$



The shaped area is $\int_a^b f$.

There are non-integrable functions

- Unbounded functions are integrable.

An example:

$$\varphi(x) = \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

- Bounded functions may be non-integrable.

$$g(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$$

g is non-integrable on any $[a,b]$.